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1987 J. Phys. A: Math. Gen. 20 293

(http://iopscience.iop.org/0305-4470/20/2/015)

J. Phys. A: Math. Gen. **20** (1987) 293-303. Printed in the UK

On the quasi-periodic solutions to the discrete non-linear Schrödinger equation

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Received 30 December 1985, in final form 15 April 1986

Abstract. We have obtained the quasi-periodic solutions to the discrete non-linear Schrödinger equation (DNLSE) by a variant of a method due to Date and Tanaka. It is shown explicitly that the non-linear field variables at the different lattice points can be determined in a recursive fashion in terms of combinations of Reimann's θ functions depending on lattice position and time.

1. Introduction

Periodic solutions to non-linear partial differential equations have attracted the attention of researchers over the last decade. Initial studies of the properties of such non-linear equations were confined to continuous versions of these equations [1]. The importance of discrete non-linear equations and their study was initiated by the pioneering paper of Ablowitz and Ladik [2], where the discrete counterpart of the AKNS system was discussed and the corresponding inverse problem for the asymptotically zero boundary condition formulated. The periodic inverse problem for discrete systems, on the other hand, has received comparatively less attention from researchers.

In the continuous case we have the well known formulation of Kreiechver [3] and Dubrovin [4], based on algebraic geometry. Two other important formalisms are those of Date and Tanaka [5] and Forest and McLaughlin [6]. An initial attempt at studying the periodic inverse problem for the Toda lattice was made by Kac and Van Moerveke [7]. Some results for the periodic spectral problem for the discrete Laplacian were also obtained by Kato [8], but the corresponding formalism is quite complicated. Here we have shown that an analogue of the formalism of references [5] and [6] can lead to an elegant formulation of the inverse problem for the discrete non-linear Schrödinger equation.

2. The DNLSE and the inverse problem

The equation under consideration is [9]

$$\mathrm{i}q_{nt} = (\Delta x)^{-2}(q_{n+1} + q_{n-1} - 2q_n) \pm q_n q_n^*(q_{n+1} + q_{n-1}). \tag{1}$$

The equations for the inverse problem are [9]

$$\begin{aligned} v_{n+1} &= F_n(z) v_n \\ \partial v_n/\partial t &= G_n(z) v_n \end{aligned} \tag{2}$$

where $F_n(z)$, $G_n(z)$ are matrices;

$$F_n(z)=\begin{pmatrix} z & q_n\Delta x \\ \mp q_n^*\Delta x & \bar{z}^{1} \end{pmatrix} \tag{3}$$

$$G_n(z)=\frac{\mathrm{i}}{(\Delta x)^2}\begin{pmatrix} 1-z^2\mp(\Delta x)^2 q_n q_{n-1}^* & \Delta x(-q_n z+\bar{z}^{1} q_{n-1}) \\ \pm\Delta x(q_{n-1}^* z-\bar{z}^{i} q_n^* & -(1-\bar{z}^2\mp(\Delta x)^2 q_{n-1}q_n^*) \end{pmatrix} \tag{4}$$

and v_n is a two-component vector (v_{1n}, v_{2n}). The consistency between (3) and (4) is equivalent to (1) and is written as

$$\partial F_n/\partial t = G_{n+1}F_n - F_nG_n. \tag{5}$$

We now assume that the non-linear field q_n obeys the periodic boundary condition $q_{n+N+1}=q_n$ with period $N+1$, where N is an arbitrary non-negative integer. The inverse problem for the periodic sine-Gordon system as formulated in [6] starts with the equations satisfied by the square eigenfunction associated with the Lax equation. We show here that if we consider the analogue of the monodromy matrix (as considered by Date and Tanaka in the continuous case) for the discrete case, then the matrix elements of the monodromy matrix do possess properties similar to those of square eigenfunctions and these properties can be properly exploited to formulate the inverse problem for the periodic case in an effective manner.

From the first equation of (2) we observe that F_n can be interpreted as the transfer matrix over the single lattice site. Let us now construct the translation operator $H_n(z)$ over the total period as

$$v_{n+N+1} = H_n(z)v_n. \tag{6}$$

Now from equation (2)

$$v_{n+N+1} = F_{n+N}v_{n+N}.$$

We may also write

$$\begin{aligned} v_{n+N} &= F_{n+N-1}v_{n+N-1} \\ &\vdots \\ v_{n+1} &= F_nv_n. \end{aligned}$$

From these we immediately deduce

$$\begin{aligned} v_{n+N+1} &= F_{n+N}F_{n+N-1}\dots F_nv_n \\ &= \left(\overset{\curvearrowleft}{\prod_{j=0}^{N}} F_{n+j}\right)v_n \end{aligned} \tag{7}$$

where the curved arrow indicates the order of increase of the indices in the product. It is to be noted that the monodromy matrix $H_n(z)$ satisfies the equations

$$\begin{aligned} H_{n+1}F_n &= F_nH_n \\ \partial H_n/\partial t &= G_nH_n - H_nG_n. \end{aligned} \tag{8}$$

The matrix H_n is such that

$$\begin{aligned} &\det H_n = \det H_m \\ &\mathrm{Sp}\, H_n = \mathrm{Sp}\, H_m \\ &(\partial/\partial t)\det H_n = 0 \\ &(\partial/\partial t)\,\mathrm{Sp}\, H_n = 0 \qquad n, m = 0, 1, 2\dots. \end{aligned} \tag{9}$$

We now propose to write $H_n(z)$ as

$$H_n = \begin{pmatrix} \phi_n + f_n & -g_n \\ h_n & \phi_n - f_n \end{pmatrix} \tag{10}$$

whence condition (8) leads to

$$\begin{aligned} z(f_{n+1} - f_n) &= \mp g_{n+1} q_n^* \Delta x + h_n q_n \Delta x \\ \bar{z}^1(f_{n+1} - f_n) &= h_{n+1} q_n \Delta x - g_n q_n^* \Delta x \\ -z g_n + \bar{z}^1 g_{n+1} &= q_n \Delta x (f_{n+1} + f_n) \\ -\bar{z}^1 h_n + z h_{n+1} &= \mp q_n^* \Delta x (f_{n+1} + f_n). \end{aligned} \tag{11}$$

Before proceeding further we demonstrate that these two definitions of $H_n(z)$ (one through equation (7) and the other with the help of (10)) are actually consistent.

Let us consider the case where the lattice period $N = 1$ (in the appendix we give results for the case where $N = 2$). Then from equation (7) we obtain

$$\begin{aligned} H_n(z) &= F_{n+1}(z) F_n(z) \\ &= \begin{pmatrix} z^2 \mp q_{n+1} q_n^* (\Delta x)^2 & z q_n \Delta x + \bar{z}^1 q_{n+1} \Delta x \\ \mp z q_{n+1}^* \Delta x \mp \bar{z}^1 q_n^* \Delta x & \mp q_n q_{n+1}^* (\Delta x)^2 + \bar{z}^2. \end{pmatrix} \end{aligned} \tag{12}$$

which should be identified with (10).

We obtain

$$\begin{aligned} \phi_n + f_n &= z^2 \mp q_{n+1} q_n^* (\Delta x)^2 \\ \phi_n - f_n &= \bar{z}^2 \mp q_n q_{n+1}^* (\Delta x)^2 \\ h_n &= \mp z q_{n+1}^* \Delta x \mp \bar{z}^1 q_n^* \Delta x \\ -g_n &= z q_n \Delta x + \bar{z}^1 q_{n+1} \Delta x. \end{aligned} \tag{13}$$

Now let us change n to $n+1$ and use the periodicity $q_{n+2} = q_n$ (for $N = 1$). This immediately leads to

$$\begin{aligned} \phi_{n+1} &= \phi_n \\ h_{n+1} &= \mp z q_n^* \Delta x \mp \bar{z}^1 q_{n+1}^* \Delta x \\ -g_{n+1} &= z q_{n+1} \Delta x + \bar{z}^1 q_n \Delta x \\ f_{n+1} - f_n &= \mp (q_n q_{n+1}^* - q_{n+1} q_n^*)(\Delta x)^2. \end{aligned} \tag{14}$$

If we now substitute the values of g_{n+1} and h_n in the first equation of (11), it reproduces $z(f_{n+1} - f_n)$ given by (14). The same conclusion is valid for other elements also. The proof is easily extended for any other values of N.

We now evaluate the time evolution of (f_n, g_n, h_n) which are given as

$$\begin{aligned} \frac{\partial f_n}{\partial t} &= \frac{\mathrm{i}}{(\Delta x)^2} [\Delta x(-q_n z + \bar{z}^1 q_{n-1}) h_n \pm \Delta x (q_{n-1}^* z - \bar{z}^1 q_n^*) g_n] \\ \frac{\partial g_n}{\partial t} &= \frac{\mathrm{i}}{(\Delta x)^2} \{2 f_n \Delta x (-q_n z + \bar{z}^1 q_{n-1}) \\ &\quad + g_n |[(2 - z^2 - \bar{z}^2) \mp (\Delta x)^2 (q_n q_{n-1}^* + q_{n-1} q_n^*)]|\}| \end{aligned} \tag{15}$$

$$\frac{\partial h_n}{\partial t} = \frac{\mathrm{i}}{(\Delta x)^2}\{\pm 2 f_n \Delta x (q_{n-1}^* z - \bar{z}^1 q_n^*) + h_n[(-2 + z^2 + \bar{z}^2) \pm (\Delta x)^2 (q_n q_{n-1}^* + q_n^* q_{n-1})]\}.$$

3. Solution of the periodic inverse problem

From the procedure laid down in the papers of Date and Tanaka, and Forest and McLaughlin it is evident that the success of the whole procedure depends crucially on the series expansion of the square eigenfunction in powers of Z (the eigenvalue).

To proceed further we note that

$$\begin{aligned} & f_{n+1}^2 - g_{n+1} h_{n+1} = f_n^2 - g_n h_n \\ & (\partial/\partial t)(f_n^2 - g_n h_n) = 0 \end{aligned} \tag{16}$$

so that $f_n^2 - g_n h_n = P(z^2)$ is independent of both n and t. An identical situation occurred in reference [6] where they showed that $f^2 - gh$ is independent of both x and t in the continuous case. Property (16) further clarifies the similarity of our set (f_n, g_n, h_n) with the square eigenfunctions of the continuous case. With these comments in mind we now represent the function $P(z^2)$ as

$$p(z^2) = \sum_{K=0}^{2N+2} P_K z^{2K} = P_{2N+2} \prod_{j=1}^{2N+2} (z^2 - E_j) \tag{17}$$

so that E_j are the zeros of $P(z^2)$. Now we try a polynomial solution for f_n, g_n, h_n in the form

$$\begin{aligned} f_n(z, t) &= \sum_{K=0}^{N+1} f_n^{(K)}(t) z^{2K} \\ g_n(z, t) &= \sum_{K=0}^{N} g_n^{(K)}(t) z^{2K+1} \\ h_n(z, t) &= \sum_{K=0}^{N} h_n^{(K)}(t) z^{2K+1}. \end{aligned} \tag{18}$$

Substituting these expansions in (11) and (15) and comparing various powers of z we obtain

$$\begin{aligned} & f_{n+1}^{(N+1)} = f_n^{(N+1)} \\ & f_{n+1}^{(0)} - f_n^{(0)} = (\Delta x)(\mp q_n^* g_{n+1}^{(0)} + q_n h_n^{(0)}). \end{aligned} \tag{19}$$

Furthermore from the equation sets (11) and (15) we can easily deduce

$$\begin{aligned} & q_n = -g_n^{(N)}/2\Delta x f_n^{(N+1)} \\ & g_{n+1}^{(0)} = \Delta x q_n (f_{n+1}^{(0)} + f_n^{(0)}). \end{aligned} \tag{20}$$

Also by equating the coefficients of z^{2N+2} we obtain

$$q_{n-1}^* = \mp h_n^{(N)}/2\Delta x f_n^{(N+1)}$$

along with (21)

$$-h_n^{(0)} = \mp q_n^* \Delta x (f_{n+1}^{(0)} + f_n^{(0)}).$$

Utilising (19)-(21) we can deduce

$$f_{n+1}^{(0)}=f_n^{(0)}. \tag{22}$$

At this stage it is not out of place to comment that equations for q_n and q_{n-1}^* in (20) and (21) will serve as the base of the inverse problem once the functions $g_n^{(N)}$, $h^{(N)}$, $f_n^{(N+1)}$ are known from some other sources.

4. Further properties of the set (f_n, g_n, h_n)

Let us now consider the expansions of the functions $f_n^*(z^*, t)$ and $f_n(\bar{z}^1, t)$ in the form

$$f_n^*(z^*, t)=\sum_{K=0}^{N+1} f_n^{*(K)} z^{2K}$$

$$f_n(\bar{z}^1, t)=\sum_{K=0}^{N+1} f_n^{(K)} \bar{z}^{2K}.$$

It is evident that

$$f_n^*(z^*, t)=-c(z) f_n(\bar{z}^1, t) \tag{23}$$

with

$$c(z)=z^{2(N+1)} \qquad f_n^{*(K)}=-f_n^{(N+1-K)}. \tag{24}$$

We now postulate the existence of zeros of the function $g_n(z, t)$ in the complex z plane and write it in the following form:

$$\begin{aligned} g_n(t, z) &= z g_n^{(N)} \prod_{j=1}^{N} (z^2-\mu_j(n, t)) \\ &= \sum_{K=0}^{N} g_n^{(K)} z^{2K+1}. \end{aligned} \tag{25}$$

From the two relations of (25) we have immediately

$$g_n^{(0)}=g_n^{(N)}(-1)^N \prod_{j=1}^{N} \mu_j(n, t) \tag{26}$$

along with

$$g_n^{(0)}=2\Delta x q_{n-1} f_n^{(0)}.$$

Now from the complex conjugate of equation (21) we have

$$2\Delta x q_{n-1}=\mp h_n^{(N)*}/f_n^{(N+1)*}.$$

Hence

$$\begin{aligned} g_n^{(0)} &= \mp(h_n^{(N)*}/f_n^{(N+1)*}) f_n^{(0)} \\ &= g_n^{(N)}(-1)^N \prod_{j=1}^{N} \mu_j(n, t) \end{aligned}$$

or

$$\pm \frac{f_n^{(0)}}{f_n^{(N+1)*}} \frac{h_n^{(N)*}}{g_n^{(N)}}=(-1)^{N+1} \prod_{j=1}^{N} \mu_j(n, t). \tag{27}$$

Using (20) and (21) we obtain

$$\frac{q_{n-1}}{q_n} = \pm \frac{f_n^{(N+1)}}{f_n^{(N+1)*}} \frac{h_n^{(N)*}}{g_n^{(N)}}$$

so finally we obtain

$$\frac{q_{n-1}}{q_n} = (-1)^{N+1} \frac{f_n^{(N+1)}}{f_n^{(0)}} \prod_{j=1}^{N} \mu_j(n, t)$$

or

$$q_{n-1} = q_n(-1)^{N+1} \frac{f_n^{(N+1)}}{f_n^{(0)}} \prod_{j=1}^{N} \mu_j(n, t). \tag{28}$$

Equation (28) is of central importance in our inversion scheme. Since this formula suggests that once the position of the zeros $\mu_j(n, t)$ are known as functions of n and t then the lattice field at the nth position obeys a recursion relation in terms of those at the $(n-1)$th position and $\mu_j(n, t)$.

5. Equation of motion for $\mu_j(n, t)$

To deduce the equation which governs the behaviour of $\mu_j(n, t)$ we start by equating coefficients of z^{2N+1} on both sides of the second equation of (15) to obtain

$$\frac{\partial g_n^{(N)}}{\partial t} = \frac{\mathrm{i}}{(\Delta x)^2} \{2\Delta x q_{n-1} f_n^{(N+1)} - 2\Delta x q_n f_n^{(N)}$$
$$+ g_n^{(N)}[2 \mp (\Delta x)^2(q_n q_{n-1}^* + q_{n-1} q_n^*)] - g_n^{(N-1)}\}. \tag{29}$$

We then use the following hierarchy of relations to simplify equation (29):

$$g_n^{(N)} = -2f_n^{(N+1)} q_n \Delta x \tag{30a}$$

$$g_n^{(N-1)} = -g_n^{(N)} \left(\sum_{j=1}^{N} \mu_j \right) = 2\Delta x f_n^{(N+1)} q_n \left(\sum_{j=1}^{N} \mu_j \right) \tag{30b}$$

$$(f_n^{(N+1)})^2 = P_{2N+2} \tag{30c}$$

$$2f_n^{(N+1)} f_n^{(N)} - g_n^{(N)} h_n^{(N)} = P_{2N+1} \tag{30d}$$

$$f_n^{(N)} / f_n^{(N+1)} = \mp 2q_n q_{n-1}^* (\Delta x)^2 + P_{2N+1}/2P_{2N+2} \tag{30e}$$

which immediately leads to

$$-2f_n^{(N+1)} \frac{\partial q_n}{\partial t} \Delta x$$
$$= \frac{\mathrm{i}}{(\Delta x)^2} \Bigg(2\Delta x q_{n-1} f_n^{(N+1)} - 2\Delta x q_n f_n^{(N)}$$
$$- 2f_n^{(N+1)} q_n \Delta x [2 \mp (\Delta x)^2 (q_n q_{n-1}^* + q_{n-1} q_n^*] - 2\Delta x f_n^{(N+1)} q_n \sum \mu_j \Bigg)$$

and finally can be converted to the form

$$\frac{\partial}{\partial t} I_n q_n = \frac{\mathrm{i}}{(\Delta x)^2}\Bigg(2 \pm 3|q_n|^2(\Delta x)^2(-1)^N \frac{f_n^{(N+1)*}}{f_n^{(0)*}} \prod_{j=1}^{N} \mu_j^*(n,t)$$
$$+\frac{P_{2N+1}}{2P_{2N+2}} + (-1)^N \frac{f_n^{(N+1)}}{f_n^{(0)}} \prod_{j=1}^{N} \mu_j(n,t) + \sum_{j=1}^{N} \mu_j(n,t)$$
$$\pm |q_n|^2(\Delta x)^2(-1)^N \frac{f_n^{(N+1)}}{f_n^{(0)}} \prod_{j=1}^{N} \mu_j(n,t)\Bigg). \tag{31}$$

We now use the original non-linear difference equation with n shifted by one:

$$\mathrm{i}\frac{\partial q_{n-1}}{\partial t} = \frac{q_n + q_{n-2} - 2q_{n-1}}{(\Delta x)^2} \pm q_{n-1} q_{n-1}^*(q_n + q_{n-2}).$$

Equating the two expressions for $\partial q_{n-1}/\partial t$ we arrive at

$$q_n + q_{n-1}\sum_{j=1}^{N} \mu_j(n,t) = \pm q_{n-1}[3q_{n-1}q_{n-2}^*(\Delta x)^2$$
$$- q_{n-1}^* q_n (\Delta x)^2 \mp P_{2N+1}/2P_{2N+2}]. \tag{32}$$

Equation (32) will also be useful for the inversion problem. We now use equation (25) in the equation for $g_n(z,t)$ to obtain

$$z\frac{\partial g_n^{(N)}}{\partial t}\prod_{j=1}^{N}(z^2-\mu_j) + z g_n^{(N)} \sum_{l=1}^{N}\prod_{j\neq l}^{N}(z^2-\mu_j)\left(-\frac{\partial \mu_l}{\partial t}\right)$$
$$= \frac{\mathrm{i}}{(\Delta x)^2}\Bigg(2\Delta x f_n(-q_n z + \bar{z}^1 q_{n-1})$$
$$+ z g_n^{(n)} \prod_{j=1}^{N}(z^2-\mu_j)[2 - z^2 - \bar{z}^2 \mp (q_n q_{n-1}^* + q_{n-1} q_n^*)]\Bigg).$$

Setting $z^2 = \mu_j$ we obtain

$$\frac{\partial \mu_j}{\partial t} = \frac{\mathrm{i}}{(\Delta x)^2}(P(\mu_j))^{1/2}\left(f_n^{(N+1)}\prod_{i\neq j}(\mu_j - \mu_l)\right)^{-1}\left(-1 + (-1)^{N+1}\frac{f_n^{(N+1)}}{f_n^{(0)}}\prod_{l\neq j}\mu_l\right). \tag{33}$$

Equation (33) gives a set of non-linear ordinary differential equations for the motion of the zeros $(\mu_j(n,t))$. This flow can be straightened out by exploiting the fact that each μ_j resides on the Riemann surface of

$$R^2(E) = \prod_{j=1}^{2N+2}(E - E_j). \tag{34}$$

We define N Abelian differentials of the first kind on it by

$$\mathrm{d}u_\nu = \frac{C_{\nu 1}E^{N-1} + \ldots + C_{\nu N}}{R(E)}\mathrm{d}E \qquad \nu = 1, 2, \ldots, N.$$

The matrix of the constants $C_{\nu\mu}$ are fixed in terms of $\{E_j\}$ and the normalisation conditions

$$\oint_a \mathrm{d}u_\nu = \delta_{\nu\mu}$$

while the 'b' cycle defines the period matrix:

$$B_{\mu\nu} = \int_b \mathrm{d}u_\nu.$$

On the Riemann surface designated in this manner, equation (33) can be written as

$$\frac{\partial \mu_j}{\partial t} = \frac{\mathrm{i}}{(\Delta x)^2}\left(-1 + (-1)^{N+1}\frac{f_n^{(N+1)}}{f_n^{(0)}}\prod_{l\neq j}\mu_l\right)\left(\prod_{K=1}^{2N+2}(\mu_j - E_K)^{1/2}\right)\left(\prod_{l\neq j}(\mu_j - \mu_l)\right)^{-1}. \tag{35}$$

We now follow the traditional path to define some functional on this Riemann surface through

$$\begin{aligned} l_j(\mu) &= -\sum_{K=1}^{N}\int_0^{\mu_K}\mathrm{d}u_j \\ &= -\sum_{l=1}^{N} C_{jl}\sum_{K=1}^{N}\int_0^{\mu_K}\frac{E^{l-1}}{R(E)}\,\mathrm{d}E \end{aligned} \tag{36}$$

from which we deduce

$$\begin{aligned} \frac{\mathrm{d}l_j}{\mathrm{d}t}(\mu) = -\frac{\mathrm{i}}{(\Delta x)^2}\sum_{l=1}^{N} C_{jl}\sum_{K=1}^{N}&\left(-1 + (-1)^{N+1}\frac{f_n^{(N+1)}}{f_n^{(0)}}\prod_{m\neq K}\mu_m\right) \\ &\times \mu_K^{l-1}\left(\prod_{n\neq K}(\mu_K - \mu_n)\right)^{-1}. \end{aligned} \tag{37}$$

With the help of standard Lagrange interpolation formulae it can be seen that these summations over μ_i are really very simple. Instead of the general case we indicate here the result for two μ, μ_1 and μ_2, where we obtain

$$\begin{aligned} \frac{\mathrm{d}l_j}{\mathrm{d}t}(\mu_1, \mu_2) = \frac{\mathrm{i}}{(\Delta x)^2} C_{j1}&\left[\left(1 + \frac{f_n^{(3)}}{f_n^{(0)}}\mu_2\right)\frac{\mu_1}{\mu_1 - \mu_2} + \left(1 + \frac{f_n^{(3)}}{f_n^{(0)}}\mu_1\right)\frac{\mu_2}{\mu_2 - \mu_1}\right] \\ + \frac{\mathrm{i}}{(\Delta x)^2} C_{j2}&\left[\left(1 + \frac{f_n^{(3)}}{f_n^{(0)}}\mu_2\right)\frac{1}{\mu_1 - \mu_2} + \left(1 + \frac{f_n^{(3)}}{f_n^{(0)}}\mu_1\right)\frac{1}{\mu_2 - \mu_1}\right] \end{aligned}$$

for the case $N = 2$. $f_n^{(3)}/f_n^{(0)} = -1$ (as shown in the appendix). Thus

$$\frac{\mathrm{d}l_j}{\mathrm{d}t}(\mu_1, \mu_2) = \frac{\mathrm{i}}{(\Delta x)^2}(C_{j1} + C_{j2}).$$

Hence we have a linear flow given as

$$l_j(\mu_1, \mu_2) = \frac{\mathrm{i}}{(\Delta x)^2}(C_{j1} + C_{j2})t + l_j^{(0)}. \tag{38}$$

Similar considerations hold also for $l_j(\mu_1, \mu_2 \ldots, \mu_N)$ and we have

$$l_j(\mu_1, \mu_2 \ldots, \mu_N) = \frac{\mathrm{i}}{(\Delta x)^2}[C_{j1} + (-1)^N C_{jN}]t + l_j^{(0)}. \tag{39}$$

Now the solutiuon for the μ_j can be obtained by the well known Jacobi inverse problem.

6. Explicit form of the solution

From our previous discussions we observe that the periodic non-linear field at the nth lattice point defined to be q_n satisfies the elegant recursion relation (28) with $\mu_j(n, t)$ satisfying (33).

There is also another algebraic connection between $\mu_i(n, t)$, q_n and q_{n-1} given by equation (32). All these equations involve some symmetric functions of the zeros $\mu_j(n, t)$. It is interesting to note that these symmetric functions of $\mu_i(n, t)$ can be expressed in terms of combinations of Riemann θ functions. Since this result is now quite standard [5] we will quote the result and use it in the following.

It is known that

$$\prod_{j=1}^{N} \mu_j(n, t) = \ln \frac{\theta(\alpha t + \gamma + \delta)\theta(\alpha t + \gamma - \delta)}{\theta(\alpha t + \gamma + \delta)\theta(\alpha t + \gamma - \sigma)} + \sum_{j=1}^{N} \oint_{a_j} \ln z \, \mathrm{d}u_j(z) \tag{40}$$

$$\sum_{j=1}^{N} \mu_j(n, t) = \frac{\partial}{\partial \tau} \ln \frac{\theta(\alpha t + \gamma + \sigma + \tau\beta)}{\theta(\alpha t + \gamma - \sigma + \tau\beta)}\bigg|_{\tau=0} + \sum_{j=1}^{N} \oint z \, \mathrm{d}u_j(z) \tag{41}$$

where the vectors α, γ, δ, σ, etc, are defined as

$$\begin{aligned} &\delta_j = u_j(0) \qquad &&\gamma_j = \sum_{i=1}^{N} u_j(\mu_i, 0) + \tfrac{1}{2} \sum_{i=1}^{N} B_{ij} - \tfrac{1}{2} \\ &\sigma_j = u_j(\infty) \qquad &&u_j = \int \mathrm{d}u_j \, \mathrm{d}z \\ &B_{ij} = \oint_{b_i} \mathrm{d}u_j(z) \qquad &&\alpha_j = C_{j,0}/f_{n_0}. \end{aligned} \tag{42}$$

We now set

$$\sum_{j=1}^{N} \mu_j(n, t) = \chi \qquad \frac{f_n^{(N+1)}}{f_n^{(0)}} \prod_{j=1}^{N} \mu_j(n, t) = \psi.$$

Using these with equation (32) we can eliminate q_{n-1} and solve for $|q_n|^2$ as

$$|q_n|^2 = \pm \frac{(-1)^{N+1}\psi^{-1} + \chi + \beta^1}{(-1)^{N+1}(\Delta x)^2 \psi^* (3|\psi|^2 - 1)}$$

where

$$\beta^1 = P_{2N+1}/2P_{2N+2}.$$

It is useful to remember that this formula is valid for N-phase periodic waves in general.

7. Discussion

In the formalism presented above we have tried to obtain the solution to the periodic inverse problem of non-linear difference equations. Our approach is to harness the similarity between the monodromy matrix (defined through the transfer matrix at each lattice site) and the square eigenfunctions defined in the continuous case. We have discussed in detail the special situations for the lattice periods $N = 1$ and $N = 2$ and have shown that the general properties can be ascertained in these particular cases. The expansion of the elements of the monodromy matrix in terms of the eigenvalue parameter Z explicitly determines the motion of the zeros $\mu_i(n, t)$ in time. By the help of the Abelian mapping such flows can be straightened out and the whole problem reduces to the solution of an Abelian inversion of the elliptic or theta functions, as in the continuous situations.

Acknowledgment

In the original version of this paper there were some mistakes and the authors are grateful to the referee for his constructive-criticism and for pointing out these defects.

Appendix. Properties of the functions $f_n^{(K)}(t)$, $g_n^{(K)}(t)$ and $h_n^{(K)}(t)$

The following relations are seen to hold:

$$
\begin{aligned}
f_n^*(z^*, t) &= -z^{2(N+1)} f_n(\bar{z}^1, t) \\
g_n^*(z^*, t) &= \pm z^{2(N+1)} h_n(\bar{z}^1, t) \\
h_n^*(z^*, t) &= \pm z^{2(N+1)} g_n(\bar{z}^1, t).
\end{aligned}
$$

Using the expansions (18) equating the coefficients from the above equations, it is easy to obtain

$$
\begin{aligned}
f_n^{(K)*}(t) &= -f_n^{(N+1-K)}(t) \\
g_n^{(K)*}(t) &= \pm h_n^{(N-K)}(t) \\
h_n^{(K)*}(t) &= \pm g_n^{(N-K)}(t).
\end{aligned}
$$

Let us illustrate the above properties for the case $N = 2$. In this case

$$H_n(z) = F_{n+2}(z) F_{n+1}(z) F_n(z)$$

or

$$
H_n(z) = \begin{pmatrix}
\begin{aligned} z^3 &\mp z q_{n+1} q_n^* (\Delta x)^2 \mp z q_{n+1}^* q_{n+2} (\Delta x)^2 \\ &\mp \bar{z}^1 q_{n+2} q_n^* (\Delta x)^2 \end{aligned} & \begin{aligned} z^2 q_n \Delta x &+ q_{n+1} \Delta x + \bar{z}^2 q_{n+2} \Delta x \\ &\mp q_n q_{n+1}^* q_{n+2} (\Delta x)^3 \end{aligned} \\
\begin{aligned} &\mp q_{n+2}^* z^2 \Delta x + q_n^* q_{n+2}^* (\Delta x)^3 \\ &\mp q_{n+1}^* \Delta x \mp \bar{z}^2 q_n^* \Delta x \end{aligned} & \begin{aligned} &\mp z q_n q_{n+2}^* (\Delta x)^2 \mp \bar{z}^1 q_{n+1} q_{n+2}^* (\Delta x)^2 \\ &+ \bar{z}^3 \mp \bar{z}^1 q_n q_{n+1}^* (\Delta x)^2 \end{aligned}
\end{pmatrix}.
$$

If we identify $z^3 H_n$ as the monodromy matrix we have

$$
\begin{aligned}
f_n &= -\tfrac{1}{2} \pm \tfrac{1}{2} z^2 (q_n q_{n+1}^* + q_{n+1} q_{n+2}^* + q_n q_{n+2}^*) \Delta x^2 \\
&\qquad \mp \tfrac{1}{2} z^4 (q_{n+1} q_n^* + q_{n+1}^* q_{n+2} + q_n^* q_{n+2}) (\Delta x)^2 + \tfrac{1}{2} z^6 \\
g_n &= -z q_{n+2} \Delta x - z^3 [q_{n+1} \Delta x \mp q_n q_{n+1}^* q_{n+2} (\Delta x)^3] - z^5 q_n \Delta x \\
h_n &= \mp z q_n^* \Delta x \mp z^3 [q_{n+1}^* \Delta x \mp q_n^* q_{n+1} q_{n+2}^* (\Delta x)^3] \mp z^5 q_{n+2}^* \Delta x
\end{aligned}
$$

so that

$$
\begin{aligned}
f_n^{(0)} &= -\tfrac{1}{2} \\
f_n^{(1)} &= \pm \tfrac{1}{2} (q_n q_{n+1}^* + q_{n+1} q_{n+2}^* + q_n q_{n+2}^*) \Delta x^2 \\
f_n^{(2)} &= \mp \tfrac{1}{2} (q_{n+1} q_n^* + q_{n+1}^* q_{n+2} + q_n^* q_{n+2}) \Delta x^2 \\
f_n^{(3)} &= \tfrac{1}{2}.
\end{aligned}
$$

Satisfying these properties, we have $f_n^{(K)*}(t) = -f_n^{(N+1-K)}(t)$ and it is trivial to show that $\partial_t f_n^{(N+1)} = 0$. Similarly, satisfying two other properties we have $g_n^{(K)*} = \pm h_n^{(N-K)}(t)$ and $h_n^{(K)*} = \pm g_n^{(N-K)}(t)$.

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